Quantum Physics 2

Exam August 26, 2011. Tentamenhal 03, 9.00-12.00.

- ◇ *Write your name and student number on each sheet you use.*
- ◇ *The exam has 4 problems.*
- ◇ *Read the problems carefully and give complete and readable answers.*
- ◇ *No books or personal notes are allowed.*

Exercise 1 *[30 points]*

i) *[5 points]*

Calculate the expectation value of the operator $\frac{1}{2}(L_x L_y - L_y L_x)$ on the state Y_l^m (which is the common eigenstate of L_z and L^2).

ii) *[5 points]*

Two $p-$electrons are in the state $|lml_1l_2\rangle = |1,-1,1,1\rangle$. What are the possible outcomes of a measurement of L_{1z} and what is their probability?

iii) *[5 points]*

Is the matrix $\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ a valid density matrix? Motivate your answer.

iv) *[5 points]*

Show that the three Pauli matrixes are projectors, i.e. $\sigma_i = \sigma_i^2$

v) *[5 points]*

A molecule is in the rotational state

$$\frac{5Y_1^1 + 4Y_7^3 + 3Y_7^1}{\sqrt{50}}.$$

What are the possible outcomes of a measurement of $\hat{L}$ en $\hat{L}_z$ and what are the respective probabilities of these outcomes?

vi) *[5 points]*

Consider an operator A acting on angular momentum space with the property $[J_z, A] = C$, where $C \neq 0$ is another operator.
Prove that $\langle j, m_j \,|[J_z, A]|\, j, m_j\rangle = 0$.

Exercise 2 *[20 points]*

A two-level system is described by the hamiltonian

$$H = \hbar\mu \begin{pmatrix} 0 & \frac{i+1}{\sqrt{2}} \\ \frac{1-i}{\sqrt{2}} & 0 \end{pmatrix}.$$

At time $t = 0$ the system is described by the state $|\phi_0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

i) *[5 points]*
Calculate eigenvalues and eigenvectors of the hamiltonian.

ii) *[5 points]*
Rewrite the initial ($t = 0$) state as an expansion of the eigenvectors.

iii) *[5 points]*
Calculate the probability to find the system, at time $t > 0$, in the state $|\phi_f\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

iv) *[5 points]*
Calculate the expectation value at time $t > 0$ of the operator

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Exercise 3 *[20 points]*

Let us consider an atomic nucleus of charge q and mass m and spin $S = \frac{3}{2}$ subjected to an highly anisotropic electrostatic field, described by the potential:

$$\phi(\mathbf{r}) = \frac{V}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2}\right)$$

i) *[5 points]*

Write down the Hamiltonian H_0 for the nucleus and find its exact eigenvalues. What is the TOTAL degeneracy of the ground state?

Hint: Try to define appropriate frequencies so that you can use a harmonic oscillator Hamiltonian with of mass m and frequency ω which in one dimension is given by ω: $H_{harm} = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2$.

ii) *[5 points]*

Write down the ground state wavefunction in coordinate representation, then evaluate the quadrupole moment $Q = \langle GS\,|3z^2 - r^2|\,GS\rangle$ of the state.

Hint: For the 1D Harmonic oscillator the normalized ground state wavefunction is $|\,\Psi_0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$.

Moreover, the following integral might be useful: $\int_{-\infty}^{+\infty} dx\, x^2 \exp(-\alpha x^2) = \frac{\sqrt{\pi}}{2\alpha^{3/2}}$.

In such an anisotropic electric field, the ground state of the nucleus is affected by the electric quadrupole interaction, which mixes the spatial and spin degrees of fredom of the system. To the lowest order, the quadrupole interaction Hamiltonian depends on the second derivative of the potential ϕ and is given by:

$$H_{\text{quad}} = \frac{qQ}{2S(S-1)\hbar}\left[\left.\frac{\partial^2\phi}{\partial x^2}\right|_0 S_x^2 + \left.\frac{\partial^2\phi}{\partial y^2}\right|_0 S_y^2 + \left.\frac{\partial^2\phi}{\partial z^2}\right|_0 S_z^2\right]$$

where Q is the quadrupole moment evaluated on the ground state.

iii) *[4 points]*

Write down the interacting Hamiltonian $H' = H_0 + H_{\text{quad}}$ and argue that $[H_0, H_{\text{quad}}] = 0$.

iv *[4 points]*

Find the *exact* energy eigenvalues of $H' = H_0 + H_{\text{quad}}$. Discuss how the non-interacting ground state (solution of H_0) splits and indicate their degeneracy.

iv) *[2 points]*

What happens to the splitting of the ground state when $a = b$? Motivate your answer.

Excercise 4 *[20 points]*

A particle with mass m resides in a 1 dimensional potential

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \text{ en } x > a \\ \beta \sin^2(\pi x/a) & \text{for } 0 \leq x \leq a \end{cases}$$

i) *[2 points]*
This problem can be seen as a 'standard' problem with a perturbation. Give the Hamilton operator for the 'standard' problem and for the perturbation.

ii) *[3 points]*
Give the wavefunctions and eigenvalues for the unperturbed system.

iii) *[4 points]*
Calculate the first order correction to the energy of the eigenstates due to the presence of the perturbation.

iv) *[3 points]*
Under which *two* conditions is the answer to the previous question a good approximation to the true eigenvalues of the problem?

v) *[4 points]*
Calculate the second order correction to the energy of the eigenstates due to the presence of the perturbation.

vi) *[4 points]*
Suppose that β is time dependent with $\beta(t) = 2\cos(\omega t)$, what are then the selection rules for transitions due to this time dependent perturbation?

Remark:

$$\int_0^1 \sin^2(\pi x)\sin(n\pi x)\sin(m\pi x)dx = \begin{cases} \frac{1}{8}(2+\delta_{n1}) & \text{for } n = m \text{ and } n, m \in \mathbb{N} \\ -\frac{1}{8} & \text{for } n = m \pm 2 \text{ and } n, m \in \mathbb{N} \\ 0 & \text{for all other cases with } n, m \in \mathbb{N} \end{cases}$$